SOLUTION OF NONLINEAR CONTACT PROBLEMS FOR A HALF-PLANE

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We will examine the problem of the pressure of a finite number of rigid dies on the boundary L of a nonlinearly elastic half-plane made of a material of the harmonic type [1, 2]. The sections of the boundary outside the dies are acted upon by specified normal loads (additional loads) [3]. Shear stresses are absent everywhere on L, as are forces and rotation at infinity. Certain contact problems for a half-plane without an additional load were examined in [4].

1. Let the physical region we are considering S^- occupy the bottom part of the plane S with the variable z = x + iy. We will designate the boudnary of S^- as L. Rigid dies press against segments $[a_kb_k]$ of the straight line L without friction, while specified normal stresses with the intensity $N_k(x)$ act on the sections $[c_kd_k]$ $(c_k < d_k,\ d_k \leqslant a_k,\ c_{k+1} \geqslant b_k)$, k = 1, 2, ..., n. We will assume that each additional load has a finite principal vector (0, $-N_0$) and satisfies the Hölder condition (including at an infinitely distant point if it occupies an infinite interval of the boundary L).

We introduce the notation: $L_1=\sum\limits_{k=1}^{n}[a_kb_k],\ L_2=\sum\limits_{k=1}^{n}[c_kd_k],\ L_3=L\diagdown(L_1+L_2)$. Then the boundary conditions of the problem take the form [5]

$$X_y^-=0 \text{ on } L,\ Y_y=-N_k(x) \text{ on } [c_k d_k],\ Y_y^-=0 \text{ on } L_3,\qquad v^-=f(x^*)+c(x^*) \text{ on } L_1, \tag{1.1}$$

where Y_y and X_y are components of the tensor of the Cauchy true stresses; $N_k(x)$ is a known function which is real on $[c_kd_k]$ and satisfies the above conditions; v^- is the normal elastic displacement of points of the line L_1; $f(x^*)$ is a real function assigned on the deformed line ($x^* = x + u^-(x)$, $u^-(x)$ is the horizontal elastic displacement of points of L_1) and characterizing the geometry of the profile of the base of the dies ($f'(x^*) \in H$); $c(x^*)$ is a piecewise-constant function; if the dies are not rigidly connected, then $c(x^*) = c_k$ on $[a_kb_k]$, while $c(x^*) = c_0$ = const on L_1 if they are rigidly connected. In the first case, we additionally assign the principal vector of the external forces acting on each die (0, $-P_k$), while in the second case we assign the principal vector (0, $-P_0$) of the acting forces.

It should be noted that the function $f(x^*) = f[x + u(x)] = f_1(x)$ assigned on L_1 is unknown, since in our (nonlinear) formulation the geometry (form) of the boundary being deformed is unknown. It should be sought in the course of solution of the problem (i.e., it is necessary to find the function u = u(x) on L_1).

To solve the problem, we use the complex representations [2, 4]

$$X_x+Y_y+4\mu=\frac{(\lambda+2\mu)\,q\Omega(q)}{\sqrt{I}},\ Y_y-X_x-2iX_y=-\frac{4(\lambda+2\mu)}{\sqrt{I}}\frac{\Omega(q)}{q}\frac{\partial z^*}{\partial z}\frac{\partial z^*}{\partial \bar{z}}; \tag{1.2}$$

$$u+iv=\frac{\mu}{\lambda+2\mu}\int\varphi'^2(z)\,dz+\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)}{\overline{\varphi'(z)}}+\overline{\psi(z)}\right]-z,\ z^*=z+u+iv, \tag{1.3}$$

where

$$\frac{\partial z^*}{\partial z}=\frac{\mu}{\lambda+2\mu}\varphi'^2(z)+\frac{\lambda+\mu}{\lambda+2\mu}\frac{\varphi'(z)}{\overline{\varphi'(z)}},\ \frac{\partial z^*}{\partial \bar{z}}=-\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)\,\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}}-\overline{\psi'(z)}\right]; \tag{1.4}$$

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$$\sqrt{I}=\frac{\partial z^*}{\partial z}\frac{\partial \bar{z}^*}{\partial \bar{z}}-\frac{\partial z^*}{\partial \bar{z}}\frac{\partial \bar{z}^*}{\partial z},\quad q=2\left|\frac{\partial z^*}{\partial z}\right|,\quad \Omega(q)=q-\frac{2(\lambda+\mu)}{\lambda+2\mu}. \tag{1.5}$$

It has been proven that

$$\varphi'(z)\neq 0 \text{ in } S^-+L. \tag{1.6}$$

In the above relations, λ and μ are the Lame constants; $\varphi(z)$ and $\psi(z)$ are functions of the complex argument $z = x + iy$ which are analytic in the region S^- and, given sufficiently large $|z|$, have the form

$$\varphi(z)=-\frac{(\lambda+2\mu)(X+iY)}{4\pi\mu(\lambda+\mu)}\ln z+z+O(1)+\text{const}, \tag{1.7}$$
$$\psi(z)=\frac{(\lambda+2\mu)(X-iY)}{2\pi\mu(\lambda+\mu)}\left[\frac{1}{2\varphi'(z)}-1\right]\ln z+O(1)+\text{const}$$

((X, Y) is the principal vector of all of the external forces applied to L).

In the absence of stresses at infinity, it folows from the first equaity of condition (1.1) that

$$\overline{\varphi(x)}\,\varphi''(x)-\varphi'^2(x)\,\psi'(x)=0 \text{ on } L. \tag{1.8}$$

Making allowance for this relation in (1.2), we obtain

$$Y_y^-=N(x)=2\mu(\lambda+\mu)\left[|\varphi'^2(x)|-1\right]/\left[\lambda+\mu+\mu|\varphi'^2(x)|\right] \text{ on } L. \tag{1.9}$$

After some elementary reductions, we find the formula

$$\ln\varphi'^2(x)+\ln\overline{\varphi'^2(x)}=2F_0(x) \text{ on } L, \tag{1.10}$$

where

$$F_0(x)=\ln\left[\frac{\lambda+\mu}{\mu}\,\frac{2\mu+N(x)}{2(\lambda+\mu)-N(x)}\right].$$

It is evident that, given safe loading conditions, $F_0(x) > 0$ in the region L.

We now differentiate Eq. (1.3) with respect to x and take (1.8) into account in the resulting relation. Then, after performing certain obvious transformations,

$$\ln\varphi'^2(x)-\ln\overline{\varphi'^2(x)}=2i\delta(x) \text{ on } L \tag{1.11}$$

($\delta(x) = \arctan(dv/dx^*)$ is a real function on L).

We introduce the new function $\Omega(z) = \ln\varphi'^2(z)$. In accordance with (1.6), this function will be analytic in the region S^-.

Assuming th time $\delta(x)$ to be known, we write the boundary conditions as

$$\operatorname{Im}\Omega(x)=\delta(x) \text{ on } L_1,\quad \operatorname{Re}\Omega(x)=F_0(x) \text{ on } L_2+L_3. \tag{1.12}$$

Here, $\delta(x)$ and $F_0(x)$ satisfy the Hölder condition on L_1 and $L_2 + L_3$, respectively (in the latter case, including an infinitely distant point).

In the above relations (as in (1.10) and (1.11)), by $\ln\varphi'^2(x)$ we mean the boundary value (from S^- on L) of the holomorphic (in S^-, following (1.6)) function $\ln\varphi'^2(z)$. This will be the value for the branch for which $\lim_{z\to\infty}\Omega(z) = 0$.

Thus, to determine the function $\Omega(z)$, holomorphic in S^- and vanishing at infinity, we obtain the familiar mixed problem for a half-plane – the Keldysh–Sedov problem. The solution of the class h_0 of this problem has the form [6]

$$\Omega(z)=\frac{1}{\pi X(z)}\left[\int_{L_1}\frac{X(x)\delta(x)dx}{x-z}-i\int_{L_2}\frac{X(x)F_0(x)dx}{x-z}+P_{n-1}(z)\right], \tag{1.13}$$

where X(z) is the canonical function of this class in the corresponding contact problem

$$X(z) = \sqrt{(z-a_1)(z-b_1)\dots(z-a_n)(z-b_n)}.$$

By X(z), we mean the branch which is holomorphic on a plane S cut out along L_1 so that

$$\lim_{z\to\infty}[z^{-n}X(z)] = 1. \tag{1.14}$$

Also, $X(x) \equiv X^+(x)$, i.e., X(x) is the boundary value taken by the function X(z) on the left side of L_1. In (1.13)

$$P_{n-1}(z) = C_1 z^{n-1} + C_2 z^{n-2} + \dots + C_n \tag{1.15}$$

is an arbitrary polynomial with imaginary constants that are determined during the solution of the problem. In particular, in accordance with (1.7), (1.13-1.15), one of them – the constant C_1 – is determined as

$$C_1 = -(\lambda + 2\mu)iY/2\pi\mu(\lambda+\mu). \tag{1.16}$$

This means that we have

$$\varphi'(z) = \exp[(1/2)\Omega(z)] \tag{1.17}$$

($\Omega(z)$ is the right side of Eq. (1.13)).

The (unique) solution (relative to $\varphi'(z)$) of problem (1.12) of the class $h(a_1, b_1, \dots, a_n, b_n)$ takes the form

$$\varphi'(z) = \exp\left[\frac{X(z)}{2\pi}\int_{L_1}\frac{\delta(x)dx}{X(x)(x-z)} + \frac{X(z)}{2\pi i}\int_{L_2}\frac{F_0(x)dx}{X(x)(x-z)}\right] \tag{1.18}$$

with observation of the following solvability conditions:

$$\int_{L_1}\frac{x^n\delta(x)dx}{X(x)} - i\int_{L_2}\frac{x^nF_0(x)dx}{X(x)} = -\frac{(\lambda+2\mu)iY}{2\mu(\lambda+\mu)}$$

and

$$\int_{L_1}\frac{x^k\delta(x)dx}{X(x)} = i\int_{L_2}\frac{x^kF_0(x)dx}{X(x)} \quad \text{at} \quad k = 0, 1, \dots, n-1.$$

Finally, we present the (unique) solution of the class $h(a_1, a_2, \dots, a_n)$ (relative to $\varphi'(z)$):

$$\varphi'(z) = \exp\left[\frac{X_a(z)}{2\pi X_b(z)}\left(\int_{L_1}\frac{X_b(x)\,\delta(x)\,dx}{X_a(x)\,(x-z)} - i\int_{L_2}\frac{X_b(x)\,F_0(x)\,dx}{X_a(x)\,(x-z)} + C_1\right)\right], \tag{1.19}$$

where

$$\begin{aligned} X_a(z) &= \sqrt{(z-a_1)(z-a_2)\dots(z-a_n)}, \\ X_b(z) &= \sqrt{(z-b_1)(z-b_2)\dots(z-b_n)}, \end{aligned} \tag{1.20}$$

while, in accordance with (1.7), (1.20), the constant C_1 is determined from the formula

$$C_1 = \int_{L_1}\frac{X_b(x)\,\delta(x)\,dx}{X_a(x)} - i\int_{L_2}\frac{X_b(x)\,F_0(x)\,dx}{X_a(x)} - \frac{(\lambda+2\mu)\,iY}{2\mu\,(\lambda+\mu)}.$$

After we find the function $\varphi(z)$, we can find the other sought potential $\psi(z)$ from (1.8) by using a Cauchy integral. As can easily be shown, with allowance for (1.7) we have

$$\psi'(z) = -\frac{1}{2\pi i}\int_L\frac{\overline{\varphi(x)}\,\varphi''(x)dx}{\varphi'^2(x)\,(x-z)} - \frac{(\lambda+2\mu)iY}{4\pi\mu\,(\lambda+\mu)}\left[\frac{\varphi''(z)\ln z}{\varphi'^2(z)} - \frac{1}{2\pi i}\int_L\frac{\varphi''(x)\ln x\,dx}{\varphi'^2(x)(x-z)}\right].$$

We determine the field of the elastic elements from (1.2-1.5). In particular, we calculate the normal contact stresses in accordance with (1.9), while we find the elastic displacement of points of the line L from the formula

$$u_x' + iv_x' = \left[\frac{\mu}{\lambda+2\mu} + \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{1}{|\varphi'^2(x)|}\right]\varphi'^2(x) - 1.$$

2. Example 1. Die with a Rectilinear Horizontal Base. Let there be one die (n = 1) occupying a specified segment $[ab]$ of the real axis L, while external normal stresses with a constant intensity P_1 and P_2 act on sections $[c_1d_1]$, $[c_2d_2]$ ($c_1 < d_1$, $d_1 < a$, $c_2 < d_2$, $c_2 > b$) outside the die on the same line. The die is loaded by external forces, the principal vector of which will be designated as $(0, -P_0)$. We assume that the die can move only translationally. The case when it may be inclined is discussed in the following example.

In accordance with (1.13) and (1.17), in the given problem we have

$$\varphi'(z) = \exp\left[\frac{1}{2\pi\sqrt{(z-a)(z-b)}}\left(\int_a^b \frac{\sqrt{(x-a)(x-b)}\,\delta(x)dx}{x-z} - i\int_{c_1}^{d_1}\frac{\sqrt{(x-a)(x-b)}\,F_0^{(1)}(x)dx}{x-z} - i\int_{c_2}^{d_2}\frac{\sqrt{(x-a)(x-b)}\,F_0^{(2)}(x)dx}{x-z} + C\right)\right]. \tag{2.1}$$

Here, C is a constant determined from (1.16) and the conditions of the problem in the form

$$C = (\lambda + 2\mu)(P_0 + P_1 + P_2)i/4\pi\mu(\lambda + \mu). \tag{2.2}$$

However, in the given case, $\delta(x) = 0$ on $[ab]$, while $F_0^{(1)}(x)$ and $F_0^{(2)}(x)$ are known constants. Taking this into account, after calculating the necessary integrals we find from (2.1) that

$$\varphi'(z) = \exp\left[\frac{1}{2\pi\sqrt{(z-a)(b-z)}}\left(\frac{(\lambda+2\mu)(P_0+P_1+P_2)}{4\mu(\lambda+\mu)} - F_0^{(1)}(z)R_0^{(1)}(z) - F_0^{(2)}(z)R_0^{(2)}(z)\right)\right], \tag{2.3}$$

where

$$F_0^{(i)}(z) = \ln\{[(\lambda+\mu)(2\mu+P_i)\mu^{-1}][2(\lambda+\mu)-P_i]^{-1}\} \equiv F_0^{(i)},\ i = 1, 2; \tag{2.4}$$

$$R_0^{(i)}(z) = \sqrt{(d_i-a)(d_i-b)} - \sqrt{(c_i-a)(c_i-b)} + \left(z - \frac{a+b}{2}\right)\ln\left|\frac{\sqrt{(d_i-a)(d_i-b)}+2d_i-a-b}{\sqrt{(c_i-a)(c_i-b)}+2c_i-a-b}\right| + \sqrt{(z-a)(b-z)}\times$$
$$\times \ln\frac{\left[2\sqrt{(c_i-a)(c_i-b)(z-a)(b-z)}+2(z-a)(b-z)+(c_i-z)(2z-a-b)\right](d_i-z)}{\left[2\sqrt{(d_i-a)(d_i-b)(z-a)(b-z)}+2(z-a)(b-z)+(d_i-z)(2z-a-b)\right](c_i-z)}. \tag{2.5}$$

We now use (1.9) and (2.3-2.5) to find the normal contact stresses N(x) under the die in the following form (with $x \in [ab]$)

$$N(x) = \frac{2\mu(\lambda+\mu)\left\{\exp\frac{1}{\pi\sqrt{(x-a)(b-x)}}\left[\frac{(\lambda+2\mu)(P_0+P_1+P_2)}{2\mu(\lambda+\mu)} - F_0^{(1)}R_0^{(1)}(x) - F_0^{(2)}R_0^{(2)}(x)\right] - 1\right\}}{\lambda+\mu+\mu\exp\left\{\frac{1}{\pi\sqrt{(x-a)(b-x)}}\left[\frac{(\lambda+2\mu)(P_0+P_1+P_2)}{2\mu(\lambda+\mu)} - F_0^{(1)}R_0^{(1)}(x) - F_0^{(2)}R_0^{(2)}(x)\right]\right\}}. \tag{2.6}$$

This formula differs from its linear classical analog in two important properties. First, it depends on the elastic properties of the material. Second, the stresses in the neighborhood of the end of the die remain finite. This follows in particular from (2.6)

$$\lim_{x\to a(x\to b)} N(x) = 2(\lambda + \mu). \tag{2.7}$$

TABLE 1

x	P/μ 0,2	0,4	0,6	0,8
	$N(x)/2\mu$			
0	0,0977	0,1982	0,3017	0,4101
0,1	0,0978	0,1984	0,3021	0,4117
0,2	0,0979	0,1991	0,3036	0,4142
0,3	0,0991	0,2023	0,3086	0,4170
0,4	0,1021	0,2083	0,3179	0,4296
0,5	0,1064	0,2175	0,3322	0,4489
0,6	0,1117	0,2287	0,3494	0,4725
0,7	0,1698	0,3476	0,5291	0,7092
0,8	0,1715	0,3519	0,5369	0,7681
0,9	0,2216	0,4569	0,6946	0,9229
0,99	0,7159	1,3444	1,7164	1,8876

TABLE 2

x	P/μ 0,2	0,4	0,6	0,8
	$N(x)/2\mu$			
0	0,1358	0,2769	0,4208	0,5657
0,1	0,1372	0,2796	0,4252	0,5715
0,2	0,1406	0,2855	0,4341	0,5832
0,3	0,1445	0,2947	0,4481	0,6018
0,4	0,1510	0,3082	0,4685	0,6288
0,5	0,1601	0,3270	0,4971	0,6663
0,6	0,1717	0,3511	0,5334	0,7139
0,7	0,2399	0,4905	0,7387	0,9727
0,8	0,2465	0,5049	0,7608	1,0007
0,9	0,3421	0,6974	1,0288	1,3087
0,99	1,0676	1,7186	1,9295	1,9832

Table 1 shows values of the ratio $N(x)/2\mu$ at different points of the contact region with different values of $P/\mu(P_0 = P_1 = P_2 = P)$ and at $\lambda = \mu$, when $a = -1$, $b = 1$, $c_1 = -3$, $d_1 = -2$, $c_2 = 2$, $d_2 = 3$.

Table 2 shows values of the same ratio $N(x)/2\mu$ when the function $R_0^{(1)} \equiv 0$, i.e., when the additional load acts only on the right side of the die on the segment [2, 3].

As this data shows, in the above-described cases the action of the load outside the die significantly increases the contact stresses under the die.

Example 2. Die with an Inclined Rectilinear Base and an Additional Load. Let the angle of inclination of the base after deformation form the angle ω_0 with the positive direction of the L axis. The same loads as in the previous case act outside the die on the sections $[c_1 d_1]$ and $[c_2 d_2]$. Considering that $\delta(x) = \omega_0$ in the given case, we obtain the following from (1.19)

$$\varphi'(z) = \exp\left[\frac{i\omega_0}{2} + \frac{\pi\omega_0 z + C_0 - F_0^{(1)} R_0^{(1)}(z) - F_0^{(2)} R_0^{(2)}(z)}{2\pi\sqrt{(z-a)(b-z)}}\right],$$

where $C_0 = (\lambda + 2\mu)(P_0 + P_1 + P_2)/2\mu(\lambda + \mu)$, while $F_0^{(i)}$, $R_0^{(i)}$ are determined by Eqs. (2.4-2.5). By $\sqrt{(z-a)(b-z)}$, we mean the branch for which at large $|z|$ $\sqrt{(z-a)(b-z)} = -iz + O(1)$. It is characterized by the fact that it takes positive values on the higher side $[ab]$.

It follows from (2.2) that on $[ab]$

$$\varphi'^2(x)| = \exp\left[\left(\pi\omega_0 x + C_0 - F_0^{(1)} R_0^{(1)}(x) - F_0^{(2)} R_0^{(2)}(x)\right)/\pi\sqrt{(x-a)(b-x)}\right].$$

Inserting this expression into the right side of (1.9), we find the normal contact pressure on $[ab]$ in the form

TABLE 3

x	P/μ			
	0,2	0,4	0,6	0,8
	$N(x)/2\mu$			
0	0,0977	0,1992	0,3037	0,4102
0,1	0,1041	0,2053	0,3094	0,4155
0,2	0,1118	0,2138	0,3186	0,4254
0,3	0,1211	0,2251	0,3319	0,4405
0,4	0,1326	0,2399	0,3502	0,4627
0,5	0,1470	0,2595	0,3750	0,4922
0,6	0,1647	0,2834	0,4053	0,5287
0,7	0,2404	0,4207	0,6025	0,7807
0,8	0,2577	0,4313	0,7067	0,7994
0,9	0,3735	0,6116	0,8443	1,0609
0,99	1,1883	1,6348	1,8524	1,9432

TABLE 4

x	P/μ			
	0,2	0,4	0,6	0,8
	$N(x)/2\mu$			
0	0,1358	0,2768	0,4208	0,5657
0,1	0,1443	0,2871	0,4327	0,5789
0,2	0,1545	0,3005	0,4494	0,5985
0,3	0,1668	0,3179	0,4717	0,6253
0,4	0,1821	0,3405	0,5012	0,6612
0,5	0,2015	0,3699	0,5404	0,7089
0,6	0,2257	0,4069	0,5896	0,7687
0,7	0,3118	0,6639	0,8098	1,0378
0,8	0,3447	0,6048	0,8565	1,0875
0,9	0,4964	0,8470	1,1592	1,4125
0,99	1,4519	1,8536	1,9646	1,9915

$$N(x) = \frac{2\mu(\lambda+\mu)\left\{\exp\frac{1}{\sqrt{(x-a)(b-x)}}\times\left[\omega_0 x + \frac{(\lambda+2\mu)(P_0+P_1+P_2)}{2\pi\mu(\lambda+\mu)} - \frac{1}{\pi}\left(F_0^{(1)}R_0^{(1)}(x)+F_0^{(2)}R_0^{(2)}(x)\right)\right]-1\right\}}{\lambda+\mu+\mu\exp\frac{1}{\sqrt{(x-a)(b-x)}}\times\left[\omega_0 x + \frac{(\lambda+2\mu)(P_0+P_1+P_2)}{2\pi\mu(\lambda+\mu)} - \frac{1}{\pi}\left(F_0^{(1)}R_0^{(1)}(x)+F_0^{(2)}R_0^{(2)}(x)\right)\right]}. \tag{2.8}$$

As Eq. (2.6), Eq. (2.8) gives the distribution of the normal stresses in the contact region without singularities. In particular, it can be seen from (2.8) that Eq. (2.7) is valid. This in turn shows that plastic zones are formed at the indicated sites.

Table 3 presents values of $N(x)/2\mu$ at different points of the contact region with ω_0 = 0.1 and different values of P/μ for $\lambda = \mu$, when $a = -1$, $b = 1$, $c_1 = -3$, $d_1 = -2$, $c_2 = 2$, $d_2 = 3$, i.e., when the additional load acts on both sides of the die.

Table 4 shows values of $N(x)/2\mu$ for the same conditions and parameters but with $R_0^{(1)} \equiv 0$, i.e., when an additional load with the intensity P acts on the right side of the die.

An analysis of the above data shows that action of the additional load outside the die increases the contact stresses under the die.

3. Example 3. Die with a Rounded Base in the Case of an Additional Load. In this case, we take (with acceptable accuracy) $\delta(x) = x/R$, where R is a sufficiently large quantity. The additional load acting outside the die is the same as in the previous examples.

We find from (1.18) that

$$\varphi'(z)=\exp\left[\frac{\sqrt{(z-a)(z-b)}}{2\pi}\left(\frac{1}{R}\int_a^b\frac{x\,dx}{\sqrt{(x-a)(x-b)}(x-z)}-i\int_{L_2}\frac{F_0(x)dx}{\sqrt{(x-a)(x-b)}(x-z)}\right)\right] \tag{3.1}$$

with satisfaction of the conditions

$$\frac{1}{R}\int_{L_1}\frac{xdx}{\sqrt{(x-a)(x-b)}}=i\int_{L_2}\frac{F_0(x)\,dx}{\sqrt{(x-a)(x-b)}},\tag{3.2}$$
$$\frac{1}{R}\int_{L_1}\frac{x^2dx}{\sqrt{(x-a)(x-b)}}=i\int_{L_2}\frac{xF_0(x)\,dx}{\sqrt{(x-a)(x-b)}}-\frac{(\lambda+2\mu)P_0}{2\mu(\lambda+\mu)}.$$

After completing the corresponding calculations, we find from (3.1) that

$$\varphi'(z)=\exp\left[\frac{iz}{2R}+\frac{\sqrt{(z-a)(b-z)}}{2}\left(\frac{1}{R}+\frac{F_0^{(1)}R_1^{(1)}(z)+F_0^{(2)}R_1^{(2)}(z)}{\pi}\right)\right],\tag{3.3}$$

where

$$R_1(z)=\frac{1}{\sqrt{(z-a)(b-z)}}\ln\left[\frac{2\sqrt{(c_i-a)(c_i-b)(z-a)(b-z)}+(c_i-z)(2z-a-b)}{2\sqrt{(d_i-a)(d_i-b)(z-a)(b-z)}+(d_i-z)(2z-a-b)}\right].$$

In these relations, the parameters a and b are constants that are yet to be determined. They should be found from conditions (3.2). The first of them leads to the equation

$$F_0^{(1)}\left(\arcsin\frac{2d_1-a-b}{b-a}-\arcsin\frac{2c_1-a-b}{b-a}\right)-\tag{3.4}$$
$$-F_0^{(2)}\left(\arcsin\frac{2d_2-a-b}{b-a}-\arcsin\frac{2c_2-a-b}{b-a}\right)=\pi(a+b)/2R,$$

while after some simple calculations the second gives us

$$\pi(3a^2+3b^2-2ab)/8R+F_0^{(1)}\Big[\sqrt{(d_1-a)(d_1-b)}-\sqrt{(c_1-a)(c_1-b)}-\tag{3.5}$$
$$-\frac{a+b}{2}\arcsin\frac{2d_1-a-b}{b-a}+\frac{a+b}{2}\arcsin\frac{2c_1-a-b}{b-a}\Big]+$$
$$+F_0^{(2)}\Big[\sqrt{(d_2-a)(d_2-b)}-\sqrt{(c_2-a)(c_2-b)}-\frac{a+b}{2}\arcsin\frac{2d_2-a-b}{b-a}+$$
$$+\frac{a+b}{2}\arcsin\frac{2c_2-a-b}{b-a}\Big]=\frac{(\lambda+2\mu)(P_0+P_1+P_2)}{2\mu(\lambda+\mu)}.$$

In particular, at $a=-\ell$, b = ℓ, in the absence of an additional load Eq. (3.5) becomes an identity, while (3.6) leads (after some elementary calculations) to the equation

$$l=\sqrt{(\lambda+2\mu)R(P_0+P_1+P_2)/\pi\mu(\lambda+\mu)}.$$

Let us return to Eq. (3.3). We will calculate the boundary value of the function $\varphi'(z)$ on $[ab]$ and insert the resulting expression into (1.9). Then

$$N(x)=\frac{2\mu(\lambda+\mu)\left\{\exp\left[\left(\frac{1}{R}+\frac{F_0^{(1)}R_1^{(1)}(x)+F_0^{(2)}R_1^{(2)}(x)}{\pi}\right)\sqrt{(x-a)(b-x)}\right]-1\right\}}{\lambda+\mu+\mu\exp\left[\left(\frac{1}{R}+\frac{F_0^{(1)}R_1^{(1)}(x)+F_0^{(2)}R_1^{(2)}(x)}{\pi}\right)\sqrt{(x-a)(b-x)}\right]}.$$

This formula gives the distribution of the normal stresses in the contact region. As can be seen, while changing continuously within the region, these stresses vanish at its ends.

LITERATURE CITED

1. F. John, "Plane strain problems for a perfectly elastic material of harmonic type," Communs. Pure Appl. Math., 13, No. 2 (1960).
2. A. I. Lur'e, Nonlinear Theory of Elasticity, Nauka, Moscow (1980).

3. E. A. Kuznetsov, "Plane contact problem with allowance for an additional load applied outside the die," Prikl. Mekh., 18, No. 5 (1982).
4. L. G. Dobordzhginidze, "Plane contact problem of the nonlinear theory of elasticity for an elastic half-plane of a material of the harmonic type," Izv. Akad. Nauk, Mekh. Tverd. Tela, No. 4 (1987).
5. N. I. Muskhelishvili, Some Fundamental Problems of the Mathematical Theory of Elasticity [in Russian], Nauka, Moscow (1966).
6. N. I. Muskhelishvili, Singular Integral Equations [in Russian], Moscow (1968).

FUNDAMENTAL SOLUTIONS OF THE THEORY OF UNIDIRECTIONAL COMPOSITES

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In the present study, we solve a problem concerning the action of a concentrated force in an infinite undirectional composite for the two- and three-dimensional cases. The approach that is taken makes it possible to express the solution of the problem of deformation by body forces in the form of series and integrals, solve the problem of a loaded half-space (half-plane), and asymptotically obtain known solutions on fiber rupture that can be employed in crack problems.

1. We will examine an infinite unidirectional composite in which the fibers form a square grid in the section perpendicular to the reinforcement direction z. The period of the grid is H + h (the cross-sectional area of the fibers is h^2). The numbers of the nodes are represented by the subscripts j and k. The dimensionless coordinates along the fibers $\xi = z/\sqrt{Hh}$, while the dimensionless displacement $w_{j,k} = u_{j,k}/\sqrt{Hh}$. The displacement satisfies the equation [1]

$$\begin{gathered}\partial^2 w_{j,k}/\partial\xi^2 + \beta^2\Delta_{jk}w = 0, \quad -\infty < j,\ k < \infty, \\ \Delta_{jk}w = w_{j-1,k} + w_{j,k-1} - 4w_{j,k} + w_{j+1,k} + w_{j,k+1}\end{gathered} \tag{1.1}$$

and the auxiliary conditions

$$\sigma_{j,k} \to 0,\ |\xi| \to \infty,\ \sigma_{j,k} = E dw_{j,k}/d\xi; \tag{1.2}$$

$$[\sigma_z]\big|_{\xi=0} = E\left.\frac{dw_{00}}{d\xi}\right|_{\xi=+0} - E\left.\frac{dw_{00}}{d\xi}\right|_{\xi=-0} = 2Q. \tag{1.3}$$

Condition (1.3) gives a jump in the normal stress in the fiber k = j = 0 at the point $\xi = 0$. This corresponds to the application of a concentrated force $-2Qh^2$ to the fiber. Here, $\beta^2 = G/E$: E and G are the Young's modulus and shear modulus for the fiber and the binder; h and H are the width of the fiber and binder. The solution will be sought by means of double discrete Fourier transformation. Each equation of system (1.1) is multiplied by $\exp(-ijs) \times \exp(-ikq)$, $-\pi \le s, q \le \pi$ and summed over j, k within infinite limits. After completing some elementary transformations, we arrive at a linear differential equation with constant coefficients. The equation is of the second order in ξ relative to the double Fourier series:

$$w^{FF} = \sum_{j,k=-\infty}^{\infty} w_{j,k}(\xi)\exp(-ijs)\exp(-ikq). \tag{1.4}$$

The solution of (1.4) depends on two arbitrary constants. Although being independent of ξ, these constants generally depend on the parameters s and q. One of them is equal to zero, thanks to (1.2). After solving (1.4), we find $w_{j,k}$ from the inversion formula expressing

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